

## A NOTE ON HARDY SPACES AND BOUNDED OPERATORS

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**ABSTRACT.** In this note we show that if  $f \in H^p(\mathbb{R}^n) \cap L^s(\mathbb{R}^n)$ , where  $0 < p \leq 1 < s < \infty$ , then there exists a  $(p, \infty)$ -atomic decomposition which converges to  $f$  in  $L^s(\mathbb{R}^n)$ . From this fact, we prove that a bounded operator  $T$  on  $L^s(\mathbb{R}^n)$  can be extended to a bounded operator from  $H^p(\mathbb{R}^n)$  into  $L^p(\mathbb{R}^n)$  if and only if  $T$  is bounded uniformly in  $L^p$  norm on all  $(p, \infty)$ -atoms. A similar result is also obtained from  $H^p(\mathbb{R}^n)$  into  $H^p(\mathbb{R}^n)$ .

## 1. INTRODUCTION

M. Bownik in [1] gives an example of a linear functional defined on a dense subspace of Hardy space  $H^1(\mathbb{R}^n)$ , which maps all atoms into bounded scalars, but it can not be extended to a bounded functional on the whole space  $H^1(\mathbb{R}^n)$ . That example is in certain sense pathological. In [6], F. Ricci and J. Verdera show that when  $0 < p < 1$  no example like the one in [1] can be exhibited. They also studied the extension problem for operators defined on the space of finite linear combinations of  $(p, \infty)$ -atoms,  $0 < p < 1$ , and taking values in a Banach space  $B$ , and proved that if the operator is uniformly bounded on  $(p, \infty)$ -atoms, then it extends to a bounded operator from  $H^p(\mathbb{R}^n)$  into  $B$ .

The extension property for operators defined on the space of finite linear combinations of  $(1, q)$ -atoms with  $1 < q < \infty$ , and taking values in Banach spaces was studied by S. Meda, P. Sjögren and M. Vallarino in [4]. For  $0 < p \leq 1$  and  $(p, 2)$ -atoms and operators taking values in quasi-Banach spaces, by D. Yang and Y. Zhou in [8].

Y. Han and K. Zhao in [3], using the Calderón reproducing formula, give a  $(p, 2)$ -atomic decomposition for members of  $H^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ , where the composition converges also in  $L^2(\mathbb{R}^n)$  rather than only in the distributions sense. Then, using this atomic decomposition, they prove that a bounded operator  $T$  on  $L^2(\mathbb{R}^n)$  can be extended to a bounded operator from  $H^p(\mathbb{R}^n)$  into  $L^p(\mathbb{R}^n)$  if and only if  $T$  is bounded uniformly on all  $(p, 2)$ -atoms.

In this note we will show, by a simple argument, that if  $f \in H^p(\mathbb{R}^n) \cap L^s(\mathbb{R}^n)$  with  $0 < p \leq 1 < s < \infty$ , then the  $(p, \infty)$ -atomic decomposition given in [7], (Theorem 2, p. 107), whose sum converges to  $f$  in  $H^p$  norm also converges to  $f$  in  $L^s$  norm. As a consequence of this, we prove that a bounded operator  $T$  on  $L^s(\mathbb{R}^n)$  can be extended to a bounded operator from  $H^p(\mathbb{R}^n)$  into  $L^p(\mathbb{R}^n)$  if and only if  $T$  is bounded uniformly in  $L^s$  norm on all  $(p, \infty)$ -atoms. A similar result is also obtained from  $H^p(\mathbb{R}^n)$  into  $H^p(\mathbb{R}^n)$ . See Theorem 5 and Corollary 6 and 7 below.

## 2. PRELIMINARIES

For a given cube  $Q$  in  $\mathbb{R}^n$ , we denote by  $l(Q)$  its length. The following proposition gives the well known Whitney decomposition for an open nonempty proper subset of  $\mathbb{R}^n$ .

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**Proposition 1.** (See p. 463 in [2]) Let  $\mathcal{O}$  be an open nonempty proper subset of  $\mathbb{R}^n$ . Then there exists a family of closed cubes  $\{Q_k\}_k$  such that

- a)  $\bigcup_k Q_k = \mathcal{O}$  and the  $Q_k$ 's have disjoint interiors.
- b)  $\sqrt{n} l(Q_k) \leq \text{dist}(Q_k, \mathcal{O}^c) \leq 4\sqrt{n} l(Q_k)$ .
- c) If the boundaries of two cubes  $Q_i$  and  $Q_k$  touch, then

$$\frac{1}{4} \leq \frac{l(Q_i)}{l(Q_k)} \leq 4.$$

- d) For a given  $Q_k$  there exist at most  $12^n$  cubes  $Q_i$ 's that touch it.

**Remark 2.** Let  $\mathcal{F} = \{Q_k\}$  be a Whitney decomposition of a proper open subset  $\mathcal{O}$  of  $\mathbb{R}^n$ . Fix  $0 < \epsilon < \frac{1}{4}$  and denote  $Q_k^*$  the cube with the same center as  $Q_k$  but with side length  $(1 + \epsilon)$  times that of  $Q_k$ . Then  $Q_k$  touch  $Q_i$  if and only if  $Q_k^* \cap Q_i^* \neq \emptyset$ . Consequently, every point in  $\mathcal{O}$  is contained in at most  $12^n$  cubes  $Q_k^*$ . Moreover,  $0 < \epsilon < \frac{1}{4}$  can be chosen such that  $\bigcup_k Q_k^* = \mathcal{O}$ .

**Remark 3.** We consider  $\mathcal{O}_1$  and  $\mathcal{O}_2$  two open nonempty proper subset of  $\mathbb{R}^n$  such that  $\mathcal{O}_2 \subset \mathcal{O}_1$ , and for  $j = 1, 2$  let  $\mathcal{F}_j = \{Q_k^j\}$  be the Whitney decomposition of  $\mathcal{O}_j$ . Since  $\mathcal{O}_2 \subset \mathcal{O}_1$ , from Proposition 1-b, we obtain that if  $Q_i^2 \cap Q_k^1 \neq \emptyset$  then

$$l(Q_i^2) \leq 5 l(Q_k^1).$$

From this, it follows that for a given cube  $Q_i^2 \in \mathcal{F}_2$  there exist at most  $7^n$  cubes in  $\mathcal{F}_1$  such that  $Q_i^2 \cap Q_{k_j}^1 \neq \emptyset$ , for  $j = 1, \dots, 7^n$  and  $Q_i^2 \subset \bigcup_{j=1}^{7^n} Q_{k_j}^1$ ; thus  $Q_i^{2*} \subset \bigcup_{j=1}^{7^n} Q_{k_j}^{1*}$ . Finally, from Proposition 1-d and Remark 2, we obtain that there exist at most  $84^n$  cubes  $Q_k^{1*}$ 's that intersect to  $Q_i^{2*}$ .

We conclude this preliminaries with the following

**Lemma 4.** Let  $\{\mathcal{O}_j\}_{j \in \mathbb{Z}}$  be a family of subset of  $\mathbb{R}^n$  such that  $\mathcal{O}_{j+1} \subset \mathcal{O}_j$  and  $\left| \bigcap_{j=1}^{\infty} \mathcal{O}_j \right| = 0$ . Then

$$\sum_{j=-\infty}^{\infty} 2^j \chi_{\mathcal{O}_j}(x) \leq 2 \sum_{j=-\infty}^{\infty} 2^j \chi_{\mathcal{O}_j \setminus \mathcal{O}_{j+1}}(x), \quad p.p. x \in \mathbb{R}^n.$$

*Proof.* We consider  $N, M \in \mathbb{N}$ . Now a computation gives

$$\begin{aligned} \sum_{j=-N}^M 2^j \chi_{\mathcal{O}_j} &= \frac{1}{2} \sum_{j=-N+1}^M 2^j \chi_{\mathcal{O}_j} + \sum_{j=-N}^{M-1} 2^j \chi_{\mathcal{O}_j \setminus \mathcal{O}_{j+1}} + 2^M \chi_{\mathcal{O}_M} \\ &\leq \frac{1}{2} \sum_{j=-N}^M 2^j \chi_{\mathcal{O}_j} + \sum_{j=-N}^{M-1} 2^j \chi_{\mathcal{O}_j \setminus \mathcal{O}_{j+1}} + 2^M \chi_{\mathcal{O}_M}. \end{aligned}$$

Since  $\lim_{M \rightarrow \infty} \chi_{\mathcal{O}_M} = \chi_{\bigcap_{j=1}^{\infty} \mathcal{O}_j}$  and  $\left| \bigcap_{j=1}^{\infty} \mathcal{O}_j \right| = 0$  the lemma follows.  $\square$

### 3. MAIN RESULT

We recall the definition of Hardy space and  $(p, \infty)$ -atoms.

Define  $\mathcal{F}_N = \left\{ \varphi \in S(\mathbb{R}^n) : \sum_{|\beta| \leq N} \sup_{x \in \mathbb{R}^n} (1 + |x|)^N |\partial^\beta \varphi(x)| \leq 1 \right\}$ . Denote by  $\mathcal{M}$  the grand maximal operator given by

$$\mathcal{M}f(x) = \sup_{t > 0} \sup_{\varphi \in \mathcal{F}_N} |(t^{-n} \varphi(t^{-1} \cdot) * f)(x)|,$$

where  $f \in S'(\mathbb{R}^n)$  and  $N$  is a large and fix integer. For  $0 < p < \infty$ , the Hardy space  $H^p(\mathbb{R}^n)$  is the set of all  $f \in S'(\mathbb{R}^n)$  for which  $\|\mathcal{M}f\|_p < \infty$ . In this case we

define  $\|f\|_{H^p} = \|\mathcal{M}f\|_p$ .

For  $1 < p < \infty$ , it is well known that  $H^p(\mathbb{R}^n) \cong L^p(\mathbb{R}^n)$ ,  $H^1(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$  strictly, and for  $0 < p < 1$  the spaces  $H^p(\mathbb{R}^n)$  and  $L^p(\mathbb{R}^n)$  are not comparable. One of the principal interest of  $H^p(\mathbb{R}^n)$  theory is that it gives a natural extension of the results for singular integrals, originally developed for  $L^p$  ( $p > 1$ ), to the range  $0 < p \leq 1$ . This is achieved to decompose elements in  $H^p$  as sums of  $(p, \infty)$ -atoms.

For  $0 < p \leq 1$ , a  $(p, \infty)$ -atom is a measurable function  $a$  supported on a ball  $B$  of  $\mathbb{R}^n$  satisfying

- (i)  $\|a\|_\infty \leq |B|^{-\frac{1}{p}}$ ,
- (ii)  $\int x^\alpha a(x) dx = 0$ , for all multiindex  $\alpha$  with  $|\alpha| \leq n(p^{-1} - 1)$ .

Our main result is the following

**Theorem 5.** *Let  $f \in H^p(\mathbb{R}^n) \cap L^s(\mathbb{R}^n)$ , with  $0 < p \leq 1 < s < \infty$ . Then there is a sequence of  $(p, \infty)$ -atoms  $\{a_j\}$  and a sequence of scalars  $\{\lambda_j\}$  with  $\sum_j |\lambda_j|^p \leq c\|f\|_{H^p}^p$  such that  $f = \sum_j \lambda_j a_j$ , where the series converges to  $f$  in  $L^s(\mathbb{R}^n)$ .*

*Proof.* Given  $f \in H^p(\mathbb{R}^n) \cap L^s(\mathbb{R}^n)$ , let  $\mathcal{O}_j = \{x : \mathcal{M}f(x) > 2^j\}$  and let  $\mathcal{F}_j = \{Q_k^j\}_k$  be the Whitney decomposition associated to  $\mathcal{O}_j$  such that  $\bigcup_k Q_k^{j*} = \mathcal{O}_j$ . Following the proof 2.3 in [7] (p. 107-109), we have a sequence of functions  $A_k^j$  such that

- (i)  $\text{supp}(A_k^j) \subset Q_k^{j*} \cup \bigcup_{i: Q_i^{j+1*} \cap Q_k^{j*} \neq \emptyset} Q_i^{j+1*}$  and  $|A_k^j(x)| \leq c2^j$  for all  $k, j \in \mathbb{Z}$ .
- (ii)  $\int x^\alpha A_k^j(x) dx = 0$  for all  $\alpha$  with  $|\alpha| \leq n(p^{-1} - 1)$  and all  $k, j \in \mathbb{Z}$ .
- (iii) The sum  $\sum_{j,k} A_k^j$  converges to  $f$  in the sense of distributions.

From (i) we obtain

$$\sum_k |A_k^j| \leq c2^j \left( \sum_k \chi_{Q_k^{j*}} + \sum_k \chi_{\bigcup_{i: Q_i^{j+1*} \cap Q_k^{j*} \neq \emptyset} Q_i^{j+1*}} \right)$$

the remark 2 gives

$$\begin{aligned} &\leq c2^j \left( \chi_{\mathcal{O}_j} + \sum_k \sum_{i: Q_i^{j+1*} \cap Q_k^{j*} \neq \emptyset} \chi_{Q_i^{j+1*}} \right) \\ &= c2^j \left( \chi_{\mathcal{O}_j} + \sum_i \sum_{k: Q_i^{j+1*} \cap Q_k^{j*} \neq \emptyset} \chi_{Q_i^{j+1*}} \right) \end{aligned}$$

from remark 3 we have

$$\leq c2^j \left( \chi_{\mathcal{O}_j} + 84^n \sum_i \chi_{Q_i^{j+1*}} \right)$$

once again, from remark 2, we obtain

$$\leq c2^j (\chi_{\mathcal{O}_j} + \chi_{\mathcal{O}_{j+1}}) \leq c2^j \chi_{\mathcal{O}_j},$$

by Lemma 4 we can conclude that

$$\sum_{j,k} |A_k^j(x)| \leq c \sum_j 2^j \chi_{\mathcal{O}_j \setminus \mathcal{O}_{j+1}}(x), \quad p.p. x \in \mathbb{R}^n.$$

Thus

$$\begin{aligned} \int \left( \sum_{j,k} |A_k^j(x)| \right)^s dx &\leq c \sum_j \int_{\mathcal{O}_j \setminus \mathcal{O}_{j+1}} 2^{js} dx \leq c \sum_j \int_{\mathcal{O}_j \setminus \mathcal{O}_{j+1}} (\mathcal{M}f(x))^s dx \\ &\leq c \int_{\mathbb{R}^n} (\mathcal{M}f(x))^s < \infty \end{aligned}$$

since  $f \in L^s(\mathbb{R}^n)$ . Now from (iii) we obtain that the sum  $\sum_{j,k} A_k^j$  converges to  $f$  in  $L^s(\mathbb{R}^n)$ .

Finally, we set  $a_k^j = \lambda_{j,k}^{-1} A_k^j$  with  $\lambda_{j,k} = c2^j |B_k^j|^{1/p}$ , where  $B_k^j$  is the smallest ball containing  $Q_k^{j*}$  as well as all the  $Q_i^{j+1*}$  that intersect  $Q_k^{j*}$ . Then we have a sequence  $\{a_{j,k}\}$  of  $(p, \infty)$ -atoms and a sequence of scalars  $\{\lambda_{j,k}\}$  such that the sum  $\sum_{j,k} \lambda_{j,k} a_{j,k}$  converges to  $f$  in  $L^s(\mathbb{R}^n)$  with  $\sum_{j,k} |\lambda_{j,k}|^p \leq c \|f\|_{H^p}^p$  (see (38), in [7] p. 109). The proof of the theorem is therefore concluded.  $\square$

**Corollary 6.** *Let  $T$  be a bounded operator on  $L^s(\mathbb{R}^n)$  for some  $1 < s < \infty$ . Then  $T$  can be extended to a bounded operator from  $H^p(\mathbb{R}^n)$  into  $L^p(\mathbb{R}^n)$  ( $0 < p \leq 1$ ) if and only if  $T$  is bounded uniformly in the  $L^p$  norm on all  $(p, \infty)$ -atoms.*

*Proof.* Since  $T$  is a bounded operator on  $L^s(\mathbb{R}^n)$ ,  $T$  is well defined on  $H^p(\mathbb{R}^n) \cap L^s(\mathbb{R}^n)$  for  $0 < p \leq 1$ . If  $T : H^p(\mathbb{R}^n) \cap L^s(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n)$  can be extended to a bounded operator from  $H^p(\mathbb{R}^n)$  into  $L^p(\mathbb{R}^n)$ , then  $\|Ta\|_p \leq C_p \|a\|_{H^p}$  for all  $(p, \infty)$ -atom  $a$ . Since there exists a universal constant  $C$  such that  $\|a\|_{H^p} \leq C < \infty$  for all  $(p, \infty)$ -atom  $a$ ; it follows that  $\|Ta\|_p \leq C_p$  for all  $(p, \infty)$ -atom  $a$ .

Reciprocally, let  $f \in H^p(\mathbb{R}^n) \cap L^s(\mathbb{R}^n)$  by Theorem 5 there is a  $(p, \infty)$ -atom decomposition such that  $\sum_j \lambda_j a_j = f$  in  $L^s(\mathbb{R}^n)$ . Since  $T$  is bounded on  $L^s(\mathbb{R}^n)$  we have that the sum  $\sum_j \lambda_j Ta_j$  converges a  $Tf$  in  $L^s(\mathbb{R}^n)$ , thus there exists a subsequence of natural numbers  $\{j_N\}_{N \in \mathbb{N}}$  such that  $\lim_{N \rightarrow \infty} \sum_{j=-j_N}^{j_N} \lambda_j Ta_j(x) = Tf(x)$  p.p.  $x \in \mathbb{R}^n$ , this implies

$$|Tf(x)| \leq \sum_j |\lambda_j Ta_j(x)|, \quad p.p. x \in \mathbb{R}^n.$$

If  $\|Ta\|_p \leq C_p < \infty$  for all  $(p, \infty)$ -atom  $a$ , and since  $0 < p \leq 1$  we get

$$\|Tf\|_p^p \leq \sum_j |\lambda_j|^p \|Ta_j\|_p^p \leq C_p^p \sum_j |\lambda_j|^p \leq C_p^p \|f\|_{H^p}^p$$

for all  $f \in H^p(\mathbb{R}^n) \cap L^s(\mathbb{R}^n)$ . Since  $H^p(\mathbb{R}^n) \cap L^s(\mathbb{R}^n)$  is a dense subspace of  $H^p(\mathbb{R}^n)$ , the corollary follows.  $\square$

Since many operators that appear in the practice are bounded on  $L^s$  ( $1 < s < \infty$ ), in view of Corollary 6 is sufficient to prove that our operator is bounded uniformly in the  $L^p$  norm on all  $(p, \infty)$ -atoms to assure the boundedness  $H^p - L^p$ .

**Corollary 7.** *Let  $T$  be a bounded operator on  $L^s(\mathbb{R}^n)$  for some  $1 < s < \infty$ . Then  $T$  can be extended to a bounded operator on  $H^p(\mathbb{R}^n)$  ( $0 < p \leq 1$ ) if and only if  $T$  is bounded uniformly in the  $H^p$  norm on all  $(p, \infty)$ -atoms.*

*Proof.* The only if is similar to the only if of the corollary 6. By the other hand, once again by Theorem 5, for  $f \in H^p(\mathbb{R}^n) \cap L^s(\mathbb{R}^n)$ , we have a  $(p, \infty)$ -atomic decomposition such that  $\sum_j \lambda_j a_j = f$  in  $L^s$  norm. Since  $T$  is bounded on  $L^s$  we obtain

$$|\mathcal{M}(Tf)(x)| \leq \sum_j |\lambda_j \mathcal{M}(Ta_j)(x)|, \quad p.p. x \in \mathbb{R}^n.$$

Now, if  $\|Ta\|_{H^p} \leq C_p < \infty$  for all  $(p, \infty)$ -atom  $a$  with  $0 < p \leq 1$ , we get

$$\|Tf\|_{H^p}^p = \|\mathcal{M}(Tf)\|_p^p \leq \sum_j |\lambda_j|^p \|\mathcal{M}(Ta_j)\|_p^p \leq C_p^p \sum_j |\lambda_j|^p \leq C_p^p \|f\|_{H^p}^p,$$

for all  $f \in H^p(\mathbb{R}^n) \cap L^s(\mathbb{R}^n)$ . Since  $H^p(\mathbb{R}^n) \cap L^s(\mathbb{R}^n)$  is a dense subspace of  $H^p(\mathbb{R}^n)$ , the corollary follows.  $\square$

**Remark 8.** *The statement of Theorem 5 appears in [5], as well as (of implicitly way) Lemma 4 (see p. 3692).*

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